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LETTER TO THE EDITOR

Yang–Baxterization of the eight-vertex model: the braid group approach

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Abstract. The braid group representation (BGR) for the eight-vertex model is solved for the case in which the total spins are not conserved. A Yang–Baxterization scheme is applied to obtain from these BGR the spectral-dependent solution; they are found to correspond to the free-fermion and the disorder-point solutions.

The intimate relation between solvable models in statistical mechanics and the braid group representation (BGR) has become a subject of intense investigation [1, 2]. Most of these models are solved by solving their Yang–Baxter equations (YBE), which reduce to the equations for some BGR. Concerning the relationship between the spectral-dependent solutions of YBE and BGR, there are two distinct approaches: one is to reduce the known solutions of YBE to give BGR as discussed by Akutsu *et al* [3]; the other approach is to construct the solutions of YBE for given BGRs. The idea of the latter approach was proposed by Jones [4]. Recently, Ge *et al* [5] have found a systematic procedure to Yang–Baxterize a given BGR. However, the proof and applications of this scheme has been restricted to the ice-type solutions [5, 6], where the 'spin number' is conserved. In this letter, we examine this Yang–Baxterization scheme for the simplest model without such conservation—the eight-vertex model.

The braid group representation for the six-vertex model, and its higher-spin extensions, can be obtained by applying a symmetry breaking transformation to the zero-field Boltzmann weights and then take the spectral parameter u to infinity [3]. If one tries to apply the procedure to Baxter's solution for the eight-vertex model, then because the Boltzmann weights are elliptic functions in this case, sending the argument of these doubly periodic functions to infinity is ambiguous. The other apparent limit is letting u be the zeros or poles of the elliptic function; however, one regains the six-vertex-type solutions. Therefore, we solve the BGR for the eight-vertex model directly, then Yang–Baxterize it to generate the solution of YBE.

The Yang–Baxter equation in explicit form is given by

$$\sum_{\mu\nu\rho} \{\check{R}^{\mu\nu}_{\alpha\beta}(x)\check{R}^{\rho\kappa}_{\nu\gamma}(xy)\check{R}^{\lambda\omega}_{\mu\rho}(y) - \check{R}^{\mu\nu}_{\beta\gamma}(y)\check{R}^{\lambda\rho}_{\alpha\mu}(xy)\check{R}^{\omega\kappa}_{\rho\nu}(x)\} = 0 \tag{1}$$

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where $x=e^{-u}$ is the spectral parameter. The braid group representation, S, is the corresponding spectral-independent solution of (1). Let $S=\Sigma_i^N \lambda_i P_i$, where λ_i, $i=1,\ldots,N$ are the unequal eigenvalues of S, and P_i the corresponding projection matrices.

The Yang–Baxterization procedure developed in [5] is the following. Assume that the spectral-dependent solution of YBE is related to its BGR in the form

$$\check{R}(x)=\sum_{i=1}^{N}\rho_i(x)P_i \tag{2}$$

where $\rho_i(x)$ are polynomials in x of degree $N-1$ and $\rho_i(0)=\lambda_i$; hence $\check{R}(0)=S$. It has been shown that if (2) is required to satisfy the unitarity condition

$$\check{R}(x)\check{R}(x^{-1})\propto I \tag{3}$$

and the condition that the minimal polynomial for S is given by

$$\prod_{i=1}^{N}(S-\lambda_i)=0 \tag{4}$$

then $\check{R}(x)$ is uniquely determined for $N=2$ and 3:

$$\check{R}(x)=S+x\lambda_1\lambda_2S^{-1} \qquad N=2 \tag{5}$$

and

$$\check{R}(x)=\lambda_1\lambda_3x(x-1)S^{-1}+(\lambda_1+\lambda_2+\lambda_3+\lambda_1\lambda_3\lambda_2^{-1})xI-(x-1)S \qquad N=3. \tag{6}$$

The expression (5) has been proved to satisfy the YBE, assuming only the property (4). On the other hand, there is no general proof that formula (6) is a solution of the YBE, although it is the case for all the ice-type solutions known so far, for some ordering of λ_1, λ_2 and λ_3. However, there has not been an example for eight-vertex-type models.

For the eight-vertex model, $\check{R}(x)$ and S are 4×4 matrices whose elements are, in the notation of [7], the Boltzmann weights $w_1, w_2, \ldots, w_8$; explicitly,

$$\check{R}(x),\, S=\begin{array}{c} \\ ++ \\ +- \\ -+ \\ -- \end{array}\begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{pmatrix} w_1 & & & w_7 \\ & w_5 & w_3 & \\ & w_4 & w_6 & \\ w_8 & & & w_2 \end{pmatrix}\end{array}. \tag{7}$$

First, we solve for S, which satisfies (1) without the spectral parameters. Define $f^{\kappa\lambda\omega}_{\alpha\beta\gamma}$ by

$$f^{\kappa\lambda\omega}_{\alpha\beta\gamma}\equiv\sum_{\mu\nu\rho}(S^{\mu\nu}_{\alpha\beta}S^{\rho\kappa}_{\nu\gamma}S^{\lambda\omega}_{\mu\rho}-S^{\mu\nu}_{\beta\gamma}S^{\lambda\rho}_{\alpha\mu}S^{\omega\kappa}_{\rho\nu})=0. \tag{8}$$

Since, $S^{\kappa\omega}_{\alpha\beta}=0$ for $\alpha+\beta\neq\kappa+\omega \bmod 2$, assuming the convention that $\pm$ sign represents $\pm1/2$. It is easy to see that $f^{\lambda\omega\kappa}_{\alpha\beta\gamma}=0$ for $\alpha+\beta+\gamma\neq\lambda+\omega+\kappa \bmod 2$. This means half of the 64 equations of (8) are satisfied trivially; among the non-trivial equations, 12 are mere repetitions.

Consider

$$\begin{aligned} f^{+++}_{+++}&=(w_6-w_5)w_7w_8 \\ f^{++-}_{++-}+f^{++-}_{-++}&=(w_4-w_3)(w_7w_8+w_5^2). \end{aligned} \tag{9}$$

Since we are interested in the non-ice-type solutions, let $w_7w_8>0$; thus (9) implies

$$w_5=w_6 \qquad w_3=w_4. \tag{10}$$

The remaining non-trivial equations reduce to only three independent equations:

$$
\begin{aligned}
&w_5^2 - w_7 w_8 = 0 \\
&w_3^2 - w_2 w_5 + w_1 w_5 - w_1^2 = 0 \\
&w_3^2 + w_2 w_5 - w_1 w_5 - w_2^2 = 0.
\end{aligned}
\tag{11}
$$

Solving (11), we find four solutions; these are all the non-ice-type BGRs for the eight-vertex model. Among the four solutions, two are obtained from the rest by reverting the signs of w_3 and w_4; we discuss only the solutions with all weights positive.

Case I. The BGR has the form

$$
S = \begin{pmatrix} 2-t & & & p \\ & 1 & z & \\ & z & 1 & \\ p^{-1} & & & t \end{pmatrix}
\tag{12}
$$

where t and p are free parameters and $z = (t^2 - 2t + 2)^{1/2}$. The eigenvalues are $1+z$ and $1-z$; and the minimal polynomial is of degree 2. There we can substitute this in (5) and obtain the following spectral-dependent solution:

$$
\begin{aligned}
&w_1 = 2 - t + tx \qquad && w_2 = 2x + t - tx \qquad && w_3 = w_4 = (1-x)z \\
&w_5 = w_6 = 1 + x && w_7 = (1-x)p && w_8 = (1-x)/p.
\end{aligned}
\tag{13}
$$

It is easy to check that this solution satisfies the free-fermion condition $w_1 w_2 + w_3 w_4 = w_5 w_6 + w_7 w_8$, which can be solved by the Pfaffian method [8].

Case II. The BGR has the form

$$
S = \begin{pmatrix} t & & & p \\ & 1 & t & \\ & t & 1 & \\ p^{-1} & & & t \end{pmatrix}.
\tag{14}
$$

The eigenvalues are $1+t$, $1-t$ and $t-1$; and the minimal polynomial is of degree 3. Since this is consistent with (4) for $N = 3$, the candidate for $\check{R}(x)$ is given in (6). There are six ways of ordering the three eigenvalues, but because (6) is symmetric with respect to interchanging λ_1 and λ_3, there are only three unique permutations, which in the present case are

$$
\begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
1+t & 1-t & t-1 \\
1+t & t-1 & 1-t \\
1-t & 1+t & t-1.
\end{array}
\tag{15}
$$

Since two of the eigenvalues are identical except for a sign, equations (6) for the first two permutations reduce to the form (5); they give an equivalent solution, which is the following:

$$
\begin{aligned}
&w_1 = w_2 = w_3 = w_4 = (1-x)t \\
&w_5 = w_6 = 1 + x \qquad w_7 = (1+x)p \qquad w_8 = (1+x)/p.
\end{aligned}
\tag{16}
$$

The third permutation, upon substitution in (6), gives rise to the following solution:

$$\begin{array}{lll} w_1 = w_2 = t(1+x)g_1 & w_3 = w_4 = t(1-x)g_2 & \\ w_5 = w_6 = (1+x)g_2 & w_7 = (1-x)g_1 p & w_8 = (1-x)g_1/p \end{array} \tag{17}$$

where $g_1 = 1+t+x(1-t)$ and $g_2 = 1+t-x(1-t)$. Both (16) and (17) are found to satisfy the YBE by direct substitution in (1). It is worth pointing out that in general not all permutations of the three eigenvalues in (6) are solutions to the YBE [5]. Note that although (16) is effectively the $N=2$ case, but the general proof given in [5], which shows that $\check{R}(x)$ of the form (5) satisfies the YBE, assumes that the minimal polynomial of S is second degree. Since this is not true here, a direct check is still called for.

Let $p=1$; then both (16) and (17) are of the form of the Baxter model in the sense that $w_1 = w_2 = a$, $w_3 = w_4 = b$, $w_5 = w_6 = c$ and $w_7 = w_8 = d$. It is of interest to compare with the known solutions. Since (16) obviously reduces to $a=b$, $c=d$, and (17) can be shown to satisfy $a+d=b+c$, both lie in the disorder regime; furthermore, they belong to the complete-disorder solutions. Although the partition function of the eight-vertex model in these regimes can be mapped to that of a six-vertex model in the frozen ferroelectric regime [7], the mapping involves various global symmetries of the partition functions for the infinite systems. Therefore it should not be mistaken for the suggestion of any existence of mapping at the BGR levels—for a six-vertex model the BGR obeys spin conservation, but *not* for the eight-vertex model. Finally, we remark that the problem of generalizing the method of obtaining link polynomials [9] for a spin-conserved model to a non-spin-conserved one is a non-trivial open problem.

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